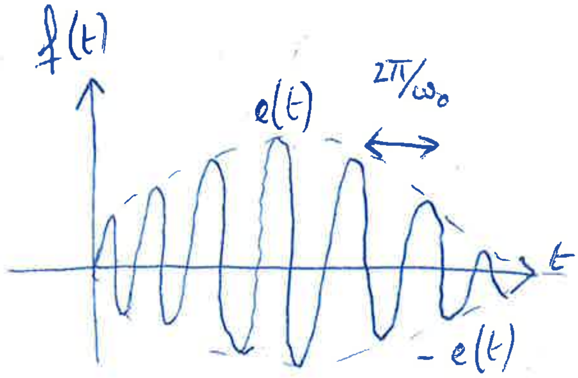
Analog differentiators

$$f(t) = e(t)\cos\omega_0 t \longrightarrow \boxed{H(\omega)} \longrightarrow g(t) = ?$$



$e(t)$ slow compared to $\frac{2\pi}{\omega_0}$, carries info

$\Rightarrow e(t) \gg \frac{de}{dt} \gg \frac{d^2e}{dt^2}$ etc ...

$H(\omega)$ transfer function / $\underset{\text{FT}}{g(\omega)} = H(\omega) f(\omega)$

Goals: 1 - Find $g(t)$

2 - Can $g(t) \sim \frac{de}{dt}$?

1 - Use Fourier transform

$$\boxed{\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega) e^{j\omega t}\, d\omega \\ f(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt \end{aligned}}$$

since $H(\omega)$ defines $g(\omega) = H(\omega) f(\omega)$.

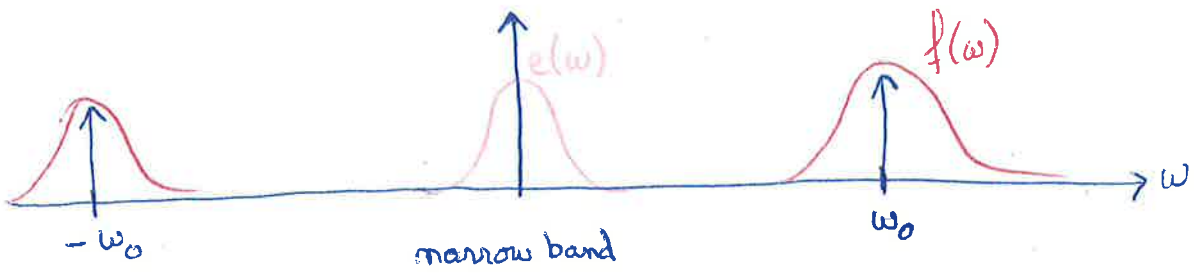
$f(\omega) = ?$

$$\mathcal{F}(\cos(\omega_0 t)) = \int_{-\infty}^{+\infty} \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt = \frac{1}{2}\overbrace{\int_{-\infty}^{+\infty} e^{j(\omega_0-\omega)t}\, dt}^{\mathcal{F}(e^{j\omega_0 t}) \overset{*}{=} 2\pi\,\delta(\omega-\omega_0)} + \frac{1}{2}\int_{-\infty}^{+\infty} e^{-j(\omega_0+\omega)t}\, dt$$

$$= \frac{1}{2}\left[\delta(\omega_0-\omega) + \delta(\omega_0+\omega)\right] \cdot 2\pi \quad = \pi\left[\delta(\omega_0-\omega) + \delta(\omega_0+\omega)\right]$$

[$*$ is because $e^{j\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(e^{j\omega_0 t})\, e^{j\omega t}\, d\omega$]

$\mathcal{F}(e(t)) = e(\omega)$



$$f(\omega) = \mathcal{F}(e(t)\cos(\omega_0 t)) = \mathcal{F}(e(t)) * \mathcal{F}(\cos \omega_0 t) = e(\omega) * \pi\left[\delta(\omega_0-\omega) + \delta(\omega_0+\omega)\right]$$

with $f * g$ being defined as $f*g = \int_{-\infty}^{+\infty} f(\Omega)\, g(\omega-\Omega)\, d\Omega$

$$\Rightarrow f(\omega) = \pi \int_{-\infty}^{+\infty} d\Omega\; e(\Omega)\left[\delta(\underset{-\Omega}{\omega_0+\omega}) + \delta(\underset{+\Omega}{\omega_0-\omega})\right]$$

$$\Rightarrow \underline{f(\omega) = \pi\left[e(\omega+\omega_0) + e(\omega-\omega_0)\right]} \quad a.$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)\, f(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2}\left\{ \int_{-\infty}^{+\infty} e(\underbrace{\omega+\omega_0}_{-\Omega}) H(\omega)\, e^{j\omega t}\, d\omega \;\; (\omega = -\omega_0 - \Omega) + \int_{-\infty}^{+\infty} e(\underbrace{\omega-\omega_0}_{\Omega}) H(\omega)\, e^{j\omega t}\, d\omega \;\; (\omega = \omega_0 + \Omega) \right\}$$

$$= \frac{1}{2}\left\{ \int_{-\infty}^{+\infty} \underbrace{H(-\omega_0-\Omega)\, e(-\Omega)}_{\left(H(\omega_0+\Omega)\, e(\Omega)\right)^*} d\Omega\; e^{-j(\omega_0 t)}\, e^{-j\Omega t} + \int_{-\infty}^{+\infty} d\Omega\; H(\omega_0+\Omega)\, e(\Omega)\, e^{j\omega_0 t}\, e^{j\Omega t} \right\}$$

$$\boxed{g(t) = \mathrm{Re}\left\{ e^{j\omega_0 t} \int_{-\infty}^{+\infty} H(\omega_0+\Omega)\, e(\Omega)\, e^{j\Omega t}\, d\Omega \right\}} \quad b.$$

just maths so far

2 - Can $g(t) \sim \frac{de}{dt}$?

Try $H(\omega) = e^{j\varphi(\omega)}$ (just a phase delay - no loss)

$$g(t) = \mathrm{Re}\left\{ e^{j\omega_0 t} \int_{-\infty}^{+\infty} e^{j\varphi(\omega_0+\Omega)} e(\Omega) e^{j\Omega t} d\Omega \right\}$$

If $e(t)$ slow enough, $e(\Omega)$ narrow band enough $\Rightarrow e^{j\varphi(\omega_0+\Omega)} \approx e^{j\varphi(\omega_0) + j\left.\frac{d\varphi}{d\omega}\right|_{\omega_0}\Omega}$

$$\Rightarrow g(t) = \mathrm{Re}\left\{ e^{j[\omega_0 t + \varphi(\omega_0)]} \underbrace{\int_{-\infty}^{+\infty} d\Omega\, e(\Omega) e^{j\Omega\left(t + \left.\frac{d\varphi}{d\omega}\right|_{\omega_0}\right)}}_{e\left(t + \left.\frac{d\varphi}{d\omega}\right|_{\omega_0}\right)} \right\}$$

def: $\tau_s = \left.\frac{d\varphi}{d\omega}\right|_{\omega_0}$ Eisenbud Wigner (Smith) scattering time

$$\boxed{g(t) = \underbrace{\cos\left(\omega_0 t + \varphi(\omega_0)\right)}_{\text{carrier phase shift } \varphi(\omega_0)} \underbrace{e\left(t + \tau_s\right)}_{\text{scattering time } \left.\frac{d\varphi}{d\omega}\right|_{\omega_0}}}$$

No derivative !

Try $H(\omega) = \underset{\in \mathbb{R}^+}{R(\omega)}\, e^{j\varphi(\omega)}$ (non unitary = radiation ~~d~~ /absorption)

$$\begin{cases} R(\omega_0+\Omega) \simeq R(\omega_0) + \left.\dfrac{dR}{d\omega}\right|_{\omega_0} \Omega \\ e^{j\varphi(\omega_0+\Omega)} \simeq e^{j\varphi(\omega_0)} \times e^{j\tau_s\Omega} \end{cases}$$

$$\Rightarrow H(\omega_0+\Omega) = e^{j\varphi(\omega_0)} \left[R(\omega_0) e^{j\tau_s\Omega} + \left.\frac{dR}{d\omega}\right|_{\omega_0} \Omega \, e^{j\tau_s\Omega} \right]$$

plug this in $g(t) = \mathrm{Re}\left\{ e^{j\omega_0 t} \int_{-\infty}^{+\infty} H(\omega_0+\Omega) e(\Omega) e^{j\Omega t} \, d\Omega \right\}$

One gets two terms:

first term: $\mathrm{Re}\left\{ e^{j(\omega_0 t+\varphi(\omega_0))} R(\omega_0) \underbrace{\int_{-\infty}^{+\infty} e^{j\Omega(t+\tau_s)} e(\Omega) \, d\Omega}_{e(t)} \right\}$

second term: $\mathrm{Re}\left\{ \underbrace{e^{j(\omega_0 t+\varphi(\omega_0))} (-j)}_{-j(c+js)=s-jc} \left.\frac{dR}{d\omega}\right|_{\omega_0} \underbrace{\int_{-\infty}^{+\infty} j\Omega \, e(\Omega) e^{j(\Omega t+\tau_s)}}_{\mathring{e}(t)} \right\}$

$$\Rightarrow g(t) = R(\omega_0)\, e(\tilde{t}) \cos(\omega_0 t + \varphi(\omega_0)) + \left.\frac{dR}{dt}\right|_{\omega_0} \mathring{e}(\tilde{t}) \cos\left(\omega_0 t + \varphi(\omega_0) - \frac{\pi}{2}\right)$$

$\underline{\tilde{t} = t - \tau_s}$

$\mathring{e}(t)$ only $\Rightarrow \begin{cases} R(\omega_0) = 0 \\ \varphi(\omega_0) = \dfrac{\pi}{2} \end{cases} \Rightarrow \boxed{H(\omega_0+\Omega) = j \left.\dfrac{dR}{d\omega}\right|_{\omega_0} \Omega \, e^{j\tau_s\Omega}}$

then $\boxed{g(t) = \left.\dfrac{dR}{d\omega}\right|_{\omega_0} \mathring{e}(t-\tau_s) \cos(\omega_0 t)}$